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The Decompositional Structure of a Generalized Hypergeometric Transformation of Convolution Type

R. K. RAINA AND M. BOLIA
Department Of Mathematics, C.T.A.E.
Campus Udaipur, Udaipur-313001
Rajasthan, India

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Abstract—In this paper, we investigate the decompositional structure of a generalized hypergeometric transformation of a convolution type. A representation of this transformation is derived in terms of the simple integral operators. Some illustrative examples, and an integral relation relating to the integral transformation are also mentioned.

Keywords—Integral transformation, Laplace transform, Weyl transform, Generalized hypergeometric function, Mellin transform, Gauss hypergeometric transformation.

1. INTRODUCTION

In terms of the Pochhammer symbol

$$(\lambda)_n = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n \in N, \end{cases} \tag{1.1}$$

the generalized hypergeometric function ${}_pF_q(z)$ is defined by [1],

$$\begin{aligned} {}_pF_q\left[\begin{matrix} a_1,\dots,a_p; \\ b_1,\dots,b_q; \end{matrix} z\right] &= {}_pF_q\left(a_1,\dots,a_p; b_1,\dots,b_q; z\right) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \cdot \frac{z^n}{n!}, \end{aligned} \tag{1.2}$$

where $p, q \in N_0 = N \cup \{o\}, p \leq q+1$; $p < q+1, |z| < \infty$; $p = q+1, |z| < 1$; and no denominator parameter equals zero or a negative integer.

A decompositional structure for the Gaussian hypergeometric transformation

$$\begin{aligned} F_c^{a,b} f(x) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \frac{1}{x} \int_0^\infty F\left(a, b; c; -\frac{t}{x}\right) f(t)\, dt \\ &= g(x), \qquad x > 0, \end{aligned} \tag{1.3}$$

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was discussed in [1]. The Fredholm type integral equation (1.3) is essentially identifiable in terms of convolution type of Mellin transforms (see [1, p. 133]). The inversion of (1.3) then follows by invertibility of the involved operators. Solutions of integral equations containing hypergeometric function in their kernels were considered among others by Higgins [2], Love [3,4]; see also Srivastava and Buschman [5]. The general idea of factorization of the convolutional type integral transformation stems from [6] wherein it has been proved that any classical integral transform of convolution type can be formally decomposed into a number of simple integral transforms.

Consider the generalized hypergeometric transformation

$$\begin{aligned} T_{(b_q)}^{(a_p)} f(x) &= T_{(b_1,\ldots,b_q)}^{(a_1,\ldots,a_p)} f(x) \\ &= \frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(b_1)\ldots\Gamma(b_q)} \int_0^\infty {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p; \\ b_1,\ldots,b_q; \end{matrix}\ -\frac{t}{x}\right] \frac{f(t)}{x}\,dt \\ &= g(x), \quad x>0, \end{aligned} \tag{1.4}$$

where $b_j \neq 0,-1,-2,\ldots$ $(j=1,\ldots,q)$; and $f \in L(0,\infty)$.

Our purpose in this paper is to consider the structure of decomposition of the generalized hypergeometric transformation (1.4). A representation of the generalized hypergeometric transformation (1.4) in terms of the simple Laplace transforms L, L^{-1}, and the Weyl operator W is derived. The inversion is easily obtained by inverting the involved integral operators. Since comprehensive tables of L, L^{-1}, and W exist (see [7,8]); therefore, our main result (Theorem 1 below) can fruitfully be applied to arrive at solutions in the theory of integral equations and integral transforms. An alternative method of seeking the solution of the integral equation (1.4) by reverting to Mellin transforms is also pointed out. We also give an integral relation involving the generalized hypergeometric transformation (1.4). Some examples are also given.

2. PRELIMINARIES AND DEFINITIONS

The Laplace transform of $g(x)$ is denoted by $L\{f(x)\}$ and is defined by

$$L\{g(t)\} = G(x) = \int_0^\infty e^{-xt} g(t)\,dt. \tag{2.1}$$

When $g(t)$ and $G(x)$ are related as above, the inverse Laplace transform of $G(x)$ is expressed as

$$L^{-1}\{G(x)\} = g(t).$$

We assume that L^{-1} is obtained with the help of the tables of the Laplace transforms or by means of the corresponding inversion formula:

$$g(t) = \frac{1}{2\pi i}\int_L e^{xt} G(x)\,dx. \tag{2.2}$$

For a complex variable s, the Mellin transform of $f(u)$ denoted by $M_s\{f(u)\}$ is defined by

$$M_s\{f(u)\} = F(s) = \int_0^\infty u^{s-1} f(u)\,du. \tag{2.3}$$

The inverse Mellin transform is given by

$$f(x) = \frac{1}{2\pi i}\int_L x^{-s} F(s)\,ds. \tag{2.4}$$

In (2.2) and (2.4), L denotes a suitable contour (see [1]). Also, the Weyl operator W^α of order α is defined by

$$W^\alpha\{f(x)\} = \frac{1}{\Gamma(\alpha)}\int_t^\infty (x-t)^{\alpha-1} f(x)\,dx, \qquad (\mathrm{Re}\ (\alpha) > 0). \tag{2.5}$$

We denoted by A the class of functions $f(x) \in L(\epsilon, E)$ for which $0<\epsilon<E<\infty$, and

$$f(x) = \begin{cases} 0\,(x^\eta), & \eta > 0, \quad x \to 0, \\ 0\,[\exp(-x^\epsilon)], & \epsilon > 0, \quad x \to \infty. \end{cases} \tag{2.6}$$

3. DECOMPOSITION IN TERMS OF L, L^{-1}, AND W

In this section, we give a representation of the generalized hypergeometric transformation (1.4) in terms of the Laplace transforms L, L^{-1}, and the Weyl operator W. We prove the following theorem.

THEOREM 1. *If*

(i) $\mathrm{Re}\,(a_i) > 1/2\,(i = 1, \dots, p)$,
(ii) $\mathrm{Re}\,(b_j) > 0\,(j = 1, \dots, q)\,(p = q + 1)$,
(iii) $f(x) \in A$,
(iv) $y^{-1/2} f(y) \in A$, *where $f(y)$ is piecewise differentiable and continuous for $y = x \geq 0$, and of bounded variation near the point $y = x$,*
(v) $M_s\{f(x)\}F(s) \in L(1/2 - i\infty, 1/2 + i\infty$, *and*
(vi) $y^{-1/2} H_{(b_q)}^{(a_p)} f \in A$ *and is of bounded variation near the point $y = x$, then, we have the following decomposition of (1.4):*

$$g(x) = x^{a_1 - 1} L\left\{x^{a_1 - b_1} L^{-1}\left\{x^{a_2 - b_1} \dots L^{-1}\left\{x^{a_{p-1} - b_{q-1}} \right.\right.\right. \\ \left.\left.\left.\times L\left\{x^{a_{p-1} - 1} L\left\{x^{a_p - 1} W^{b_q - a_p}\left\{x^{1 - b_q} f(x)\right\}\right\}\right\}\right\}\right\}\right\}. \tag{3.1}$$

PROOF. Due to Conditions (i)–(iii) and Theorem 1.2.17 of [9], $T_{(b_q)}^{(a_p)} f$ exists and belongs to $L(0, \infty)$. Then, Condition (iv) and Definition (2.4) of the inverse Mellin transform yield

$$f(x) = \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} F(s) x^{-s}\, ds. \tag{3.2}$$

Multiplying by $x^{1 - b_q}$ and applying the Weyl operator $W^{b_q - a_p}$ on both sides, we have

$$W^{b_q - a_p}\left\{x^{1 - b_q} f(x)\right\} = \frac{1}{\Gamma\left(b_q - a_p\right)} \int_t^{\infty} (x - t)^{b_q - a_p - 1} x^{1 - b_q} \\ \times \left[\frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} F(s) x^{-s}\, dx\right]. \tag{3.3}$$

We observe that along the line $s = 1/2 + i\sigma$ the real part of the power of x is $1/2 - b_q$. Thus, Conditions (i)–(iv) ensure that the double integral (3.3) is absolutely convergent, and therefore, we can change the order of integration to get

$$W^{b_q - a_p}\left\{x^{1 - b_q} f(x)\right\} = \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{\Gamma\left(a_p + s - 1\right)}{\Gamma\left(b_q + s - 1\right)} F(s) t^{1 - s - a_p}\, ds. \tag{3.4}$$

Now multiplying by $t^{a_p - 1}$, and applying the Laplace transform (2.1), we get

$$L\left\{t^{a_p - 1} W^{b_q - a_p}\left\{x^{1 - b_q} f(x)\right\}\right\} = \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} F(s) \frac{\Gamma(1 - s)\Gamma(a_p + s - 1)}{\Gamma(b_q + s - 1)} x^{s - 1}\, ds, \tag{3.5}$$

by virtue of the integral

$$\int_0^{\infty} e^{-pt} t^v\, dt = \Gamma(v + 1) p^{-v - 1}, \qquad \text{for } \mathrm{Re}\,(\nu) > -1. \tag{3.6}$$

Next, we multiply (3.5) by $x^{a_{p-1} - 1}$, and apply the Laplace transform to get

$$L\left\{x^{a_{p-1} - 1} L\left\{t^{a_p - 1} W^{b_q - a_p}\left\{x^{1 - b_q} f(x)\right\}\right\}\right\} \\ = \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} F(s) \frac{\Gamma(1 - s)\Gamma\left(a_p + s - 1\right)\Gamma\left(a_{p-1} + s - 1\right)}{\Gamma\left(b_q + s - 1\right)} t^{1 - s - a_{p-1}}\, ds. \tag{3.7}$$

Multiplication by $t^{a_{p-1}-b_{q-1}}$ on both sides and the application of the inverse Laplace transform with the help of (3.6) is seen to yield

$$L^{-1}\left\{t^{a_{p-1}-b_{q-1}}L\left\{x^{a_{p-1}-1}L\left\{t^{a_p-1}W^{b_q-a_p}\left\{x^{1-b_q}f(x)\right\}\right\}\right\}\right\}$$
$$= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{\Gamma(1-s)\Gamma(a_p+s-1)\Gamma(a_{p-1}+s-1)}{\Gamma(b_q+s-1)\Gamma(b_{q-1}+s-1)}F(s)x^{b_{q-1}+s-2}\,ds. \quad (3.8)$$

Continuing this procedure of the application of the Laplace transform and inverse Laplace transform alternately (step by step), we finally arrive at the following relation after adjustments:

$$\begin{aligned}
&x^{a_1-1}L\left\{x^{a_1-b_1}L^{-1}\left\{x^{a_2-b_1}\dots L\left\{x^{a_{p-2}-b_{q-1}}L^{-1}\left\{x^{a_{p-1}-b_{q-1}}L\left\{x^{a_p-1}\right.\right.\right.\right.\right.\right.\\
&\quad\left.\left.\left.\left.\left.\left.\times L\left\{x^{a_p-1}W^{b_q-a_p}\left\{x^{1-b_q}f(x)\right\}\right\}\right\}\right\}\right\}\right\}\right\}\\
&= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{\Gamma(1-s)\Gamma(a_p+s-1)\dots\Gamma(a_1+s-1)}{\Gamma(b_q+s-1)\dots\Gamma(b_1+s-1)}F(s)x^{-s}\,ds\\
&= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}G(s)x^{-s}\,ds\\
&= g(x).
\end{aligned} \quad (3.9)$$

This is evident because the Mellin transform of (1.4) by appealing to the formula [1, p. 302] gives

$$\begin{aligned}
M_s\left\{T_{(b_q)}^{(a_p)}f(x)\right\} &= M_s\{g(x)\}\\
&= G(s)\\
&= F(s)\frac{\Gamma(1-s)\Gamma(a_p+s-1)\dots\Gamma(a_1+s-1)}{\Gamma(b_q+s-1)\dots\Gamma(b_1+s-1)},
\end{aligned} \quad (3.10)$$

which holds when $t^{s-1}f(t) \in L(0,\infty)$, and $\max\{1-\text{Re}\,(a_i), 1-\text{Re}\,(b_j)\} < \text{Re}\,(s) < 1$, for $i = 1,\dots,p$ and $j = 1,\dots,q$ $(p = q+1)$. This completes the proof.

REMARK 1. It may be observed that in essence the decompositional structure of (1.4) given by (3.1) involves the superposition of $(2q+1)$ simple operators L, L^{-1}, and W. The solution of (1.4) can easily be obtained and represented in the form

$$\begin{aligned}
f(x) = x^{b_q-1}W^{a_p-b_q}&\left\{x^{1-a_p}L^{-1}\left\{x^{1-a_{p-1}}L^{-1}\left\{x^{b_{q-1}-a_{p-1}}\dots\right.\right.\right.\\
&\left.\left.\left.x^{b_1-a_2}L\left\{x^{b_1-a_1}L^{-1}\left\{x^{1-a_1}g(x)\right\}\right\}\right\}\right\}\right\},
\end{aligned} \quad (3.11)$$

provided that various operators involved exist.

4. SOLUTION BY MELLIN TRANSFORM

We give in this section the alternative solution of (1.4) by invoking Mellin transforms.

THEOREM 2. *If the integrals $F(s)$ and $G(s)$ defined by (2.3) exist, $f(x)$ is piecewise differentiable and continuous for $x \geq 0$, $f(x) \in A$ (where A is defined by (2.6)), $\max\{1-\text{Re}\,(a_i)\} < \text{Re}\,(s) < 1$, $\forall\, i = 1,\dots,p$, and $b_j \neq 0, -1, -2, \dots$, $\forall\, j = 1,\dots,q$, then the solution of (1.4) can be represented by*

$$f(t) = \frac{1}{2\pi i}\lim_{\sigma\to+\infty}\int_{\gamma-i\sigma}^{\gamma+i\sigma}\frac{\Pi_{j=1}^{q}\Gamma(s+b_j-1)\,G(s)}{\Pi_{i=1}^{p}\Gamma(s+a_i-1)\,\Gamma(1-s)}t^{-s}\,ds. \quad (4.1)$$

PROOF. The Mellin transform of (1.4) is given by

$$\begin{aligned}
G(s) &= M_s\left\{T_{(b_q)}^{(a_p)}f(x)\right\}\\
&= M_s\left\{\frac{\Pi_{j=1}^{p}\Gamma(a_j)}{\Pi_{j=1}^{q}\Gamma(b_j)}\int_0^\infty {}_pF_q\left[\begin{matrix}a_1,\dots,a_p; \\ b_1,\dots,b_q;\end{matrix}\; -\frac{t}{x}\right]\frac{f(t)}{x}\,dt.\right\}.
\end{aligned} \quad (4.2)$$

By applying elementary substitutions and the convolution formula [1, pp. 150,151], and the result [10, p. 727, Section 8.4.51, Entry 1], we get

$$M_s\left\{T_{(b_q)}^{(a_p)} f(x)\right\} = \frac{\Gamma(s+a_1-1)\ldots\Gamma(s+a_p-1)\Gamma(1-s)}{\Gamma(s+b_1-1)\ldots\Gamma(s+b_q-1)} F(s) = G(s). \tag{4.3}$$

Taking the inverse Mellin transform on both sides by means of [1, p. 150], we arrive at

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\sigma}^{\gamma+i\sigma} \frac{\Gamma(s+b_1-1)\ldots\Gamma(s+b_q-1)}{\Gamma(s+a_1-1)\ldots\Gamma(s+a_p-1)\Gamma(1-s)} G(s) t^{-s}\, ds, \tag{4.4}$$

where Re $(s)=\gamma$, $t>0$, which evidently leads to (4.1).

REMARK 2. Theorem 2 can also be deduced (by certain considerations applicable in the theory of fractional calculus) from [11, p. 9, Theorem 3] in which a Fredholm type integral equation involving the Fox's H-function kernel is considered.

5. SOME CONSEQUENCES OF THEOREMS 1 AND 2

In this section, we illustrate with some examples few useful deductions of Theorems 1 and 2.

EXAMPLE 1. Putting $p=2$, $q=1$; and $a_1=a$, $a_2=a-1/2$, $b_1=2a$ in (1.4); and noting the formula

$$\left\{\frac{1}{2}(1+\sqrt{1-x})\right\}^{1-2a} = {}_2F_1\left(a, a-\frac{1}{2}; 2a; x\right), \tag{5.1}$$

we have

$$g(x) = \frac{\sqrt{\pi}\,\Gamma(a-1/2)}{\Gamma(a+1/2)}\int_0^\infty x^{a-3/2}\left(\sqrt{x}+\sqrt{x+t}\right)^{1-2a} f(t)\, dt. \tag{5.2}$$

Then its solution in view of Theorem 1 (equation (3.11)) is given by

$$f(x) = x^{2a-1} W^{-a-(1/2)}\left\{x^{3/2-a} L^{-1}\left\{x^{1-a} L^{-1}\left\{x^{1-a} g(x)\right\}\right\}\right\}. \tag{5.3}$$

On the other hand, the solution of (5.2) by applying Theorem 2 is given by

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} \frac{\Gamma(s+2a-1)}{\Gamma(s+a-1)\Gamma(s+a-3/2)\Gamma(1-s)} G(s) t^{-s}\, ds \cdot \left(\frac{3}{2}-a<\text{Re }(s)<1\right). \tag{5.4}$$

EXAMPLE 2. In terms of the incomplete Beta function

$$B_x(\alpha,\beta) = \alpha^{-1} x^\alpha\, {}_2F_1(\alpha, 1-\beta; \alpha+1; x), \tag{5.5}$$

we have, on putting $q=1 (p=2)$; $a_1=\alpha$, $a_2=1-\beta$, $b_1=\alpha+1$ in (1.4),

$$g(x) = x^{\alpha-1}\Gamma(1-\beta)\int_0^\infty (-t)^{-\alpha} B_{(-t/x)}(\alpha,\beta) f(t)\, dt. \tag{5.6}$$

The solution of (5.6) by means of Theorem 1 is given by

$$f(x) = x^\alpha W^{-\alpha-\beta}\left\{x^\beta L^{-1}\left\{x^{1-\alpha} L^{-1}\left\{x^{1-\alpha} g(x)\right\}\right\}\right\}. \tag{5.7}$$

Application of Theorem 2 gives the solution of (5.6) as

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s+\alpha)}{\Gamma(s+\alpha-1)\Gamma(s-\beta)\Gamma(1-s)} G(s) t^{-s}\, ds \quad \gamma(\max\{1-\alpha,\beta\}<\text{Re }(s)<1). \tag{5.8}$$

EXAMPLE 3. In terms of the Stieltjes transform

$$S_v\{f(x)\} \int_0^\infty \frac{f(t)}{(x+t)^v}\, dt, \tag{5.9}$$

we have on putting $q = 1\,(p = 2)$ in (1.4), then with $a_1 = \nu$, $a_2 = b_1$, we have from Theorem 1 the well-known decomposition [1]:

$$g(x) = \Gamma(\nu)s_v\{f(x)\} = L\left\{x^{\nu-1}L\{f(x)\}\right\}. \tag{5.10}$$

Its solution may be written in the form

$$f(x) = L^{-1}\{x^{1-\nu}L^{-1}\{g(x)\}\}. \tag{5.11}$$

The solution of (5.10) by applying Theorem 2 is given by

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{G(s)t^{-s}\,ds}{\Gamma(s+\nu-1)\Gamma(1-s)}, \qquad (1-\nu < \operatorname{Re}(s) < 1). \tag{5.12}$$

EXAMPLE 4. If we put $q = 2$ (so that $p = 3$) in (1.4), then with $a_1 = \alpha$, $a_2 = \beta$, $a_3 = \gamma$, $b_1 = \delta$, $b_2 = \eta$, we have from Theorem 1, the following decompositional structure for the integral transformation with kernel as the Clausenian series ${}_3F_2$. If

$$\begin{aligned} g(x) &= \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\delta)\Gamma(\eta)}\int_0^\infty {}_3F_2\left(\alpha,\beta,\gamma;\delta,\eta;-\frac{t}{x}\right)\frac{f(t)}{x}\,dt \\ &= x^{\alpha-1}L\left\{x^{\alpha-\delta}L^{-1}\left\{x^{\beta-\delta}L\left\{x^{\beta-1}L\left\{x^{\gamma-1}W\left\{x^{\eta-\gamma}\left\{x^{1-\eta}f(x)\right\}\right\}\right\}\right\}\right\}\right\}, \end{aligned} \tag{5.13}$$

then its solution is given by

$$f(x) = x^{\eta-1}\left\{W^{\gamma-\eta}\left\{x^{1-\gamma}L^{-1}\left\{x^{\delta-\beta}L\left\{x^{\delta-\alpha}L^{-1}\left\{x^{1-\alpha}g(x)\right\}\right\}\right\}\right\}\right\}. \tag{5.14}$$

The solution of (5.13) by applying Theorem 2 is given by

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s+\delta-1)\Gamma(s+\eta-1)}{\Gamma(s+\alpha-1)\Gamma(s+\beta-1)\Gamma(s+\gamma-1)\Gamma(1-s)}G(s)t^{-s}\,ds, \\ (\max\{1-\alpha,1-\beta,1-\gamma\} < \operatorname{Re}(s) < 1). \tag{5.15}$$

If we put $\alpha = a$, $\gamma = b$, $\eta = c$, and set $\beta = \delta = b$, we find that (5.13) and (5.14) correspond to the pair of relations treated in [1, p. 133, equations (8.22) and (8.23)] in which the Gauss's hypergeometric transformation (1.3) is discussed.

6. INTEGRAL RELATION INVOLVING (1.4)

We establish the integral relation which is contained in the following.

THEOREM 3. *If the value $T_{(b_q)}^{(a_p)} f_i(x) = g_i(x)$ exist for $i = 1,2$ and if one of the following integrals is absolutely convergent, then*

$$\int_0^\infty f_1(t)\left\{T_{(b_q)}^{(a_p)}f_2\left(\frac{1}{t}\right)\right\}\frac{dt}{t} = \int_0^\infty f_2(t)\left\{T_{(b_q)}^{(a_p)}f_1\left(\frac{1}{t}\right)\right\}\frac{dt}{t}. \tag{6.1}$$

PROOF. Applying Mellin transformation and the convolution property, we have

$$\begin{aligned} M_s\left\{\int_0^\infty f_1(t)\left(T_{(b_q)}^{(a_p)}f_2\left(\frac{x}{t}\right)\right)\frac{dt}{t}\right\} &= F_1(s)M_s\left\{T_{(b_q)}^{(a_p)}f_2\right\} \\ &= F_1(s)F_2(s)\frac{\Gamma(s+a_1-1)\dots\Gamma(s+a_p-1)\Gamma(1-s)}{\Gamma(s+b_1-1)\dots\Gamma(s+b_q-1)} \\ &= M_s\left\{T_{(b_q)}^{(a_p)}f_1\right\}F_2(s) \\ &= M_s\left\{\int_0^\infty f_2(t)\left(T_{(b_q)}^{(a_p)}f_1\left(\frac{x}{t}\right)\right)\frac{dt}{t}\right\}. \end{aligned}$$

Noting that Mellin transform of the convolutions coincide, and the functions being transformed are continuous (since generalized hypergeometric function is analytic), the convolutions are equal, and the result follows on setting $x = 1$.

REMARK 3. By putting $p = 2$, $q = 1$, $a_1 = a$, in Theorem 3, we precisely get the relation [1, p. 133, equation (8.25)].

On specializing the parameters of the transformation (1.4), one can easily obtain the corresponding integral relations of the type (6.1) involving Laplace, Stieltjes, and Hankel transformations.

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